Calculus BC

Section 9.7 - Taylor Polynomials and Approximations

Obj: - To find Taylor and Maclaurin polynomial approximations of elementary functions.

Def: Let f be a function with derivatives of all orders throughout some interval with "c" as an interior point,

- nth **Taylor Polynomial**

$$P_n = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots$$

- nth **Maclaurin Polynomial**

$$P_n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

same as Taylor but with $c = 0$

1. Find the Taylor polynomials P_0, P_1, P_2, P_3, P_4 for the function $f(x) = \ln x$, centered at $c = 2$.

$f(x) = \ln x$ $\qquad$ $f(2) = \ln 2$

$f'(x) =$ $\qquad$ $f'(2) =$

$f''(x) =$ $\qquad$ $f''(2) =$

$f'''(x) =$ $\qquad$ $f'''(2) =$

$f^{(4)}(x) =$ $\qquad$ $f^{(4)}(4) =$

$P_0 =$

$P_1 =$

$P_2 =$

$P_3 =$

$P_4 =$

-Examine the graphs of these polynomials on a graphing calculator.

-Using P_4, approximate $f(2.5)$

- To use the Taylor polynomials to approximate $f(5.2)$, expand the polynomials so that it is centered at $c=5$

2. Find the fourth Maclaurin polynomial for $f(x)=\sin x$.
   Maclaurin series is centered at $c=$ ___

$f(x)=$ $f(0)=$

$f'(x)=$ $f'(0)=$

$f''(x)=$ $f''(0)=$

$f'''(x)=$ $f'''(0)=$

$f^{(4)}(x)=$ $f^{(4)}(0)=$

$P_4(x)=$

-Using P_4, approximate $f(.2)$

- To use the Maclaurin polynomials to approximate $f(3.2)$, consider expanding the polynomials of the function $f(x+3)=\sin(x+3)$.

-Approximation is better at x-values closer to the center c.
-Approximation is better for higher-degree polynomials than for lower-degree ones.

$$f(x) = P_n(x) + R_n(x)$$

Exact value   Approximation   Remainder

Taylor's Theorem

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$ ☺ ☺

Error = $|R_n(x)| = |f(x) - P_n(x)|$

Lagrange error bound

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$$R_n(x) = \frac{|x-c|^{n+1}}{(n+1)!} \max\left| f^{(n+1)}(z) \right|$$

$\max\left| f^{(n+1)}(z) \right|$ is the maximum value of $f^{(n+1)}(z)$ for z between x and c.

3. Find the error term for the Taylor polynomial for the approximation of $f(.2)$ using P_4 from problem 1.

4. Find the Lagrange error bound for problem 2 for $f(.2)$

Memorize the following Taylor series (centered at 0)

$\sin x$

$\cos x$

e^x

$\sin kx$

$\cos kx$

e^{kx}